Rational Behavior in Dynamic Multicriteria Games

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OUTLINE

Multicriteria dynamic game with finite horizon

Multicriteria dynamic game

The state dynamics

$$x_{t+1} = f(x_t, u_{1t}, \ldots, u_{nt}), \;\; x_0 = x\,, \tag{1}$$

where $x_t \geq 0$ – the size of the resource, $f(x_t, u_{1t}, \ldots, u_{nt})$ – natural growth function, $u_{it} \in U_i$ – the strategy of player i, $i \in N = \{1, \ldots, n\}$.

Denote $u_t = (u_{1t}, \ldots, u_{nt})$. The players' payoffs on $[0, m]$:

$$J_i = \begin{pmatrix} J_i^1 = \sum_{t=0}^{m} \delta^t g_i^1(u_t) \\ \ldots \\ J_i^k = \sum_{t=0}^{m} \delta^t g_i^k(u_t) \end{pmatrix}, \, i \in N\,, \tag{2}$$

where $g_i^j(u_t) \geq 0$ – the instantaneous payoff functions, $j = 1, \ldots, k$, $i \in N$, $\delta \in (0, 1)$ – the discount factor.

The guaranteed payoffs:

First, obtain the guaranteed payoffs (Rettieva, 2017):

1) as a solutions of zero-sum games;

2) as a solutions of zero-sum games with weighted sum of the criteria;

3) as a Nash equilibrium:

$G_1^1, \ldots, G_n^1$ – the Nash equilibrium payoffs in the dynamic game $\langle x, N, \{U_i\}_{i=1}^n, \{J_i^1\}_{i=1}^n \rangle$,

...

$G_1^k, \ldots, G_n^k$ – the Nash equilibrium payoffs in the dynamic game $\langle x, N, \{U_i\}_{i=1}^n, \{J_i^k\}_{i=1}^n \rangle$,

Multicriteria payoff functions as Nash product (guaranteed payoffs – status quo points):

$$H_1(u_{1t}, \ldots, u_{nt}) = (J_1^1(u_{1t}, \ldots, u_{nt}) - G_1^1) \cdot \ldots \cdot (J_1^k(u_{1t}, \ldots, u_{nt}) - G_1^k)\,,$$

$$\ldots$$

$$H_n(u_{1t}, \ldots, u_{nt}) = (J_n^1(u_{1t}, \ldots, u_{nt}) - G_n^1) \cdot \ldots \cdot (J_n^k(u_{1t}, \ldots, u_{nt}) - G_n^k)\,.$$

Multicriteria Nash equilibrium

Definition

Strategy profile $(u_{1t}^N, \dots, u_{nt}^N)$ is called multicriteria Nash equilibrium (Rettieva, 2017) of the problem (1), (2) if

$$H_i(u_t^N) \geq H_i(u_{1t}^N, \dots, u_{i-1\,t}^N, u_{it}, u_{i+1\,t}^N, \dots, u_{nt}^N) \ \forall u_{it} \in U_i\,,\ i \in N\,. \quad (3)$$

The presented approach guarantees that the noncooperative payoffs of the players are greater than or equal to the guaranteed ones.

Noncooperative players' payoffs:

$$J_1^N = \begin{pmatrix} J_1^{1N} = \sum_{t=0}^{m} \delta^t g_1^1(u_t^N) \\ \dots \\ J_1^{kN} = \sum_{t=0}^{m} \delta^t g_1^k(u_t^N) \end{pmatrix}, \dots, J_n^N = \begin{pmatrix} J_n^{1N} = \sum_{t=0}^{m} \delta^t g_n^1(u_t^N) \\ \dots \\ J_n^{kN} = \sum_{t=0}^{m} \delta^t g_n^k(u_t^N) \end{pmatrix}. \quad (4)$$

Rational multicriteria cooperative equilibrium

The cooperative strategies and payoffs are constructed by solving the following problem:

$$(V_1^{1c} - J_1^{1N}) \cdot \ldots \cdot (V_1^{kc} - J_1^{kN}) + \ldots + (V_n^{1c} - J_n^{1N}) \cdot \ldots \cdot (V_n^{kc} - J_n^{kN}) =$$

$$= (\sum_{t=0}^{m} \delta^t g_1^1(u_t^c) - J_1^{1N}) \cdot \ldots \cdot (\sum_{t=0}^{m} \delta^t g_1^k(u_t^c) - J_1^{kN}) + \ldots \quad (5)$$

$$+(\sum_{t=0}^{m} \delta^t g_n^1(u_t^c) - J_n^{1N}) \cdot \ldots \cdot (\sum_{t=0}^{m} \delta^t g_n^k(u_t^c) - J_n^{kN}) \to \max_{u_t^c = (u_{1t}^c, \ldots, u_{nt}^c)},$$

where J_i^{jN} – noncooperative payoffs given by (4), $i \in N$, $j = 1, \ldots, k$.

Definition

Strategy profile $u_t^c = (u_{1t}^c, \ldots, u_{nt}^c)$ is called rational multicriteria cooperative equilibrium of the problem (1), (2) if it solves the problem (5).

Dynamic stability of cooperative solution

Theorem

The solution of problem (5) satisfies the individual rationality conditions $V_i^{jc} \geq J_i^{jN}$, $i \in N$, $j = 1, \ldots, k$.

The players' cooperative payoffs for the whole game:

$$J_1^c(0) = \begin{pmatrix} J_1^{1c}(0) = \sum_{t=0}^{m} \delta^t g_1^1(u_t^c) \\ \ldots \\ J_1^{kc}(0) = \sum_{t=0}^{m} \delta^t g_1^k(u_t^c) \end{pmatrix}, \ldots, J_n^c(0) = \begin{pmatrix} J_n^{1c}(0) = \sum_{t=0}^{m} \delta^t g_n^1(u_t^c) \\ \ldots \\ J_n^{kc}(0) = \sum_{t=0}^{m} \delta^t g_n^k(u_t^c) \end{pmatrix},$$

where u_{it}^c are the cooperative strategies determined in (5).
Similarly, we determine the cooperative payoffs $J_i^c(t)$, $i \in N$, for each subgame starting from the state x_t^c at a time t.

Dynamic stability of cooperative solution

Definition

A vector

$$\beta(t) = (\beta_1(t), \ldots, \beta_n(t)),$$

where

$$\beta_1(t) = \begin{pmatrix} \beta_1^1(t) \\ \ldots \\ \beta_1^k(t) \end{pmatrix}, \ldots, \beta_n(t) = \begin{pmatrix} \beta_n^1(t) \\ \ldots \\ \beta_n^k(t) \end{pmatrix}$$

is a payoff distribution procedure (PDP) for the multicriteria dynamic game (1), (2) if

$$J_1^c(0) = \sum_{t=0}^{m} \delta^t \beta_1(t), \ldots, J_n^c(0) = \sum_{t=0}^{m} \delta^t \beta_n(t), \tag{6}$$

or in extended form,

$$\begin{cases} J_1^{1c}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_1^1(t)\,, \\ \qquad \dots \\ J_1^{kc}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_1^k(t)\,, \end{cases}, \dots, \begin{cases} J_n^{1c}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_n^1(t)\,, \\ \qquad \dots \\ J_n^{kc}(0) = \sum\limits_{t=0}^{m} \delta^t \beta_n^k(t)\,. \end{cases}$$

Definition

A vector $\beta(t) = (\beta_1(t), \dots, beta_n(t))$ is a time-consistent (Petrosyan, 1977) PDP for the multicriteria dynamic game (1), (2) if for every $t \geq 0$,

$$J_1^c(0) = \sum_{\tau=0}^{t} \delta^\tau \beta_1(\tau) + \delta^{t+1} J_1^c(t+1)\,,$$
$$\dots$$
$$J_n^c(0) = \sum_{\tau=0}^{t} \delta^\tau \beta_n(\tau) + \delta^{t+1} J_n^c(t+1)\,, \qquad (7)$$

Dynamic stability of cooperative solution

or in extended form,

$$\begin{cases} J_1^{1c}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^1(\tau) + \delta^{t+1} J_1^{1c}(t+1)\,, \\ \qquad \qquad \qquad \dots \\ J_1^{kc}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_1^k(\tau) + \delta^{t+1} J_1^{kc}(t+1)\,, \end{cases}$$

$$\dots$$

$$\begin{cases} J_n^{1c}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_n^1(\tau) + \delta^{t+1} J_n^{1c}(t+1)\,, \\ \qquad \qquad \qquad \dots \\ J_n^{kc}(0) = \sum\limits_{\tau=0}^{t} \delta^\tau \beta_n^k(\tau) + \delta^{t+1} J_n^{kc}(t+1)\,. \end{cases}$$

Dynamic stability of cooperative solution

Theorem

A vector $\beta(t) = (\beta_1(t), \ldots, \beta_2(t))$, *where*

$$\beta_1(t) = J_1^c(t) - \delta J_1^c(t+1)\,,$$
$$\ldots$$
$$\beta_n(t) = J_n^c(t) - \delta J_n^c(t+1) \qquad (8)$$

is a time-consistent payoff distribution procedure for the multicriteria dynamic game (1), (2).

Conditions for rational behavior

We adopt rationality conditions, namely irrational-behavior-proofness condition (Yeung, 2006) and each step rational behavior condition (Mazalov, Rettieva, 2010, 2012) for dynamic multicriteria games.

Definition

The multicriteria cooperative solution $J^c(t) = (J_1^c(t), \dots, J_n^c(t))$ satisfies the irrational behavior proofness condition if

$$\sum_{\tau=0}^{t} \delta^\tau \beta_i(\tau) + \delta^{t+1} J_i^N(t+1) \geq J_i^N(0) \qquad (9)$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ – time-consistent payoff distribution procedure (8) and $J_i^N(t)$ is the noncooperative payoff (4) of player i, $i \in N$.

Conditions for rational behavior

Or in extended form,

$$\begin{cases} \sum_{\tau=0}^{t} \delta^{\tau} \beta_1^1(\tau) + \delta^{t+1} J_1^{1N}(t+1) \geq J_1^{1N}(0)\,, \\ \qquad \qquad \qquad \dots \\ \sum_{\tau=0}^{t} \delta^{\tau} \beta_1^k(\tau) + \delta^{t+1} J_1^{kN}(t+1) \geq J_1^{kN}(0)\,, \end{cases}$$

$$\dots$$

$$\begin{cases} \sum_{\tau=0}^{t} \delta^{\tau} \beta_n^1(\tau) + \delta^{t+1} J_n^{1N}(t+1) \geq J_n^{1N}(0)\,, \\ \qquad \qquad \qquad \dots \\ \sum_{\tau=0}^{t} \delta^{\tau} \beta_n^k(\tau) + \delta^{t+1} J_n^{kN}(t+1) \geq J_n^{kN}(0)\,. \end{cases}$$

Conditions for rational behavior

Definition

The multicriteria cooperative solution $J^c(t) = (J_1^c(t), \dots, J_n^c(t))$ satisfies each step rational behavior condition if

$$\beta_i(t) + \delta J_i^N(t+1) \geq J_i^N(t) \tag{10}$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ – time-consistent payoff distribution procedure (8) and $J_i^N(t)$ is the noncooperative payoff (4) of player i, $i \in N$. Or in extended form,

$$\begin{cases} \beta_i^1(t) + \delta J_i^{1N}(t+1) \geq J_i^{1N}(t)\,, \\ \qquad\qquad \dots \\ \beta_i^k(t) + \delta J_i^{kN}(t+1) \geq J_i^{kN}(t)\,. \end{cases}$$

Conditions for rational behavior

The conditions for rational behavior (9) and (10) for problem (1), (2) are

$$(1-\delta^{t+1})(J_i^c(0)-J_i^N(0))+\delta^{t+1}\sum_{\tau=0}^{t}\delta^{\tau}(g_i(u_\tau^c)-g_i(u_\tau^N))\geq 0\ \forall t\,, \quad (11)$$

$$(1-\delta)(J_i^c(t)-J_i^N(t))+\delta^{t+2}(g_i(u_t^c)-g_i(u_t^N))\geq 0\ \forall t\,,\ i\in N\,, \quad (12)$$

where

$$g_i(u)=\begin{pmatrix} g_i^1(u) \\ \dots \\ g_i^k(u) \end{pmatrix}.$$

The each step rational behavior conditions is fulfilled if $g_i(u_t^c)-g_i(u_t^N)$ $\forall t$, $i\in N$, and the irrational behavior proofness condition is true if $\sum_{\tau=0}^{t}\delta^{\tau}(g_i(u_\tau^c)-g_i(u_\tau^N))$ $\forall t$, $i\in N$. Hence, the each step rational behavior condition yields the Yeung's condition.

Resource management problem with finite horizon

Bi-criteria resource management problem

n players exploit the fish stock over finite time horizon $[0, m]$.
The dynamics of the fishery is

$$x_{t+1} = \varepsilon x_t - u_{1t} - \ldots - u_{nt}\,, \;\; x_0 = x\,, \tag{13}$$

where $x_t \geq 0$ – the size of population at a time t, $\varepsilon \in (0,1)$ – natural birth rate, $u_{it} \geq 0$ – the catch of player i, $i \in N = \{1, \ldots, n\}$.
The players' net revenues over finite time horizon are

$$J_1 = \begin{pmatrix} J_1^1 = \sum_{t=0}^{m} \delta^t p_1 u_{1t} \\ J_1^2 = -\sum_{t=0}^{m} \delta^t c u_{1t}^2 \end{pmatrix}, \ldots, J_n = \begin{pmatrix} J_n^1 = \sum_{t=0}^{m} \delta^t p_n u_{nt} \\ J_n^2 = -\sum_{t=0}^{m} \delta^t c u_{nt}^2 \end{pmatrix}, \tag{14}$$

where $p_i \geq 0$ – the price of the resource for player i, $i \in N$, $c \geq 0$ – the catching cost, $\delta \in (0,1)$ – the discount factor.

$G_1^1, \ldots, G_n^1$ – Nash equilibrium in the game $\langle x, N, \{U_i\}_{i=1}^n, \{J_i^1\}_{i=1}^n \rangle$. Applying the Bellman principle for the linear form of the strategies and value functions, we obtain the Nash equilibrium strategies

$$u_{1t} = \ldots = u_{nt} = \frac{\varepsilon - 1}{n - 1} x_t \,,$$

and also the population size in this equilibrium

$$x_t = \left(\frac{n - \varepsilon}{n - 1} \right)^t x_0 \,.$$

Guaranteed payoffs

Then the guaranteed payoffs take the form

$$G_1^1 = p_1 A x_0\,, \ldots, G_n^1 = p_n A x_0,$$

where

$$A = \frac{\varepsilon - 1}{n-1} \frac{(\delta(n-\varepsilon))^{m+1} + (n-1)^{m+1}}{(n-1)^m(\delta(n-\varepsilon) - n + 1)}\,.$$

Similarly find the Nash equilibrium in the dynamic game $\langle x, N, \{U_i\}_{i=1}^n, \{J_i^2\}_{i=1}^n \rangle$:

$$G_1^2 = \ldots = G_n^2 = -cGx_0^2\,,$$

where

$$G = \Big(\frac{2n - \varepsilon^2 + \varepsilon\sqrt{4n^2 + \varepsilon^2 - 4n}}{n(-\varepsilon + \sqrt{4n^2 + \varepsilon^2 - 4n})}\Big)^2$$

$$\times \frac{(2\delta n)^{m+1} - (\varepsilon - \sqrt{4n^2 + \varepsilon^2 - 4n})^{m+1}}{(\varepsilon - \sqrt{4n^2 + \varepsilon^2 - 4n})^m(2\delta n - \varepsilon + \sqrt{4n^2 + \varepsilon^2 - 4n})}\,.$$

Multicriteria Nash equilibrium

To construct **multicriteria Nash equilibrium** we solve the next problem

$$p_1 c(\sum_{t=0}^{m} \delta^t u_{1t} - Ax)(-\sum_{t=0}^{m} \delta^t u_{1t}^2 + Gx^2) \to \max_{u_{1t}},$$

$$\dots$$

$$p_n c(\sum_{t=0}^{m} \delta^t u_{nt} - Ax)(-\sum_{t=0}^{m} \delta^t u_{nt}^2 + Gx^2) \to \max_{u_{nt}}.$$

Multicriteria Nash equilibrium

Considering the process starting from one-stage till m-stage game and seeking linear strategies $u_{it}^N = \gamma_{it}^N x_t$ we get

$$\gamma_{1t}^N = \ldots = \gamma_{nt}^N = \gamma_t^N = \frac{\varepsilon^{t-1}\gamma_1^N}{1 + n\gamma_1^N \sum_{j=0}^{t-2} \varepsilon^j}\,,\ t = 2, \ldots, m. \tag{15}$$

The players' strategy at the last stage γ_1^N is can be determined from the equation

$$\Big[3\varepsilon^{2(m-1)} \sum_{j=0}^{m-1} \delta^j - 2\varepsilon^{m-1} n \sum_{j=0}^{m-2} \varepsilon^j A - n^2 (\sum_{j=0}^{m-2} \varepsilon^j)^2 G\Big] (\gamma_1^N)^2 -$$

$$-2(\varepsilon^{m-1} A + \sum_{j=0}^{m-2} \varepsilon^j G n)\gamma_1^N - G = 0\,.$$

Multicriteria cooperative equilibrium

The noncooperative payoffs

$$J_i^{1N}(x) = \sum_{t=0}^{m-1} \delta^t p_i \gamma_{m-t}^N x\,,\ \ i \in N\,,$$

$$J_1^{2N}(x) = \ldots = J_n^{2N}(x) = -c \sum_{t=0}^{m-1} \delta^t (\gamma_{m-t}^N)^2 x^2\,.$$

To obtain the **rational multicriteria cooperative equilibrium** we solve the next problem

$$p_1(\sum_{t=0}^{m} \delta^t u_{1t}^c - Px)(-\sum_{t=0}^{m} \delta^t (u_{1t}^c)^2 + Kx^2) + \ldots \tag{16}$$

$$+p_n(\sum_{t=0}^{m} \delta^t u_{nt}^c - Px)(-\sum_{t=0}^{m} \delta^t (u_{nt}^c)^2 + Kx^2) \to \max_{u_{1t}^c,\ldots,u_{nt}^c}\,, \tag{17}$$

where $P = \sum_{t=0}^{m-1} \delta^t \gamma_{m-t}^N$, $K = \sum_{t=0}^{m-1} \delta^t (\gamma_{m-t}^N)^2$.

Multicriteria cooperative equilibrium

Considering the process starting from one-stage till m-stage game and seeking linear strategies $u^c_{it} = \gamma^c_{it} x_t$ we get

$$\gamma^c_{1t} = \ldots = \gamma^c_{nt} = \gamma^c_t = \frac{\varepsilon^{t-1}\gamma^c_1}{1 + n\gamma^c_1 \sum\limits_{j=0}^{t-2} \varepsilon^j}\,,\ t = 2, \ldots, m. \tag{18}$$

The players' strategy at the last stage γ^c_1 at the last stage can be determined from the equation

$$\Big[3\varepsilon^{2(m-1)} \sum_{j=0}^{m-1} \delta^j - 2\varepsilon^{m-1} n \sum_{j=0}^{m-2} \varepsilon^j P - n^2 (\sum_{j=0}^{m-2} \varepsilon^j)^2 K\Big] (\gamma^c_1)^2 -$$
$$-2(\varepsilon^{m-1} P + \sum_{j=0}^{m-2} \varepsilon^j K n)\gamma^c_1 - K = 0\,.$$

PDP

The time-consistent payoff distribution procedure takes the form

$$\beta_i(t) = \begin{pmatrix} \beta_i^1(t) \\ \beta_i^2(t) \end{pmatrix}, \ i = 1, \dots, n, \ t = 0, \dots, m-1,$$

where

$$\beta_i^1(t) = p_i \delta^t \gamma_{m-t}^c x_t + p_i(1-\delta) \sum_{\tau=t+1}^{m-1} \delta^\tau \gamma_{m-\tau}^c x_\tau ,$$

$$\beta_i^2(t) = -c\delta^t (\gamma_{m-t}^c)^2 x_t^2 - c(1-\delta) \sum_{\tau=t+1}^{m-1} \delta^\tau (\gamma_{m-\tau}^c)^2 x_\tau^2 , \ i = 1, \dots, n .$$

Proposition

The conditions for rational behavior in problem (13), (14) are fulfilled if $\gamma_1^c \geq \gamma_1^N$.

Modelling

$$m = 15,\ n = 5,\ \varepsilon = 1.3,\ p_1 = \ldots = p_5 = 100,\ c = 50,\ \delta = 0.8.$$

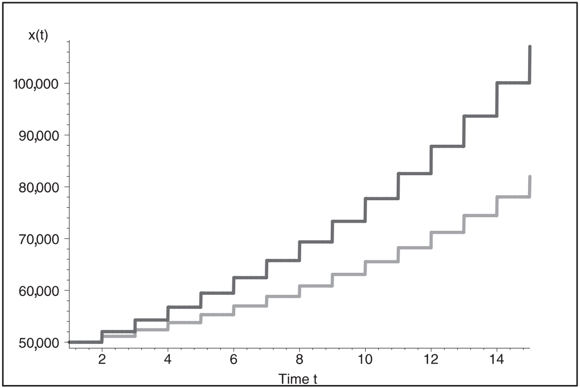

Fig. 1. Population dynamics: cooperation and Nash equilibrium

Modelling

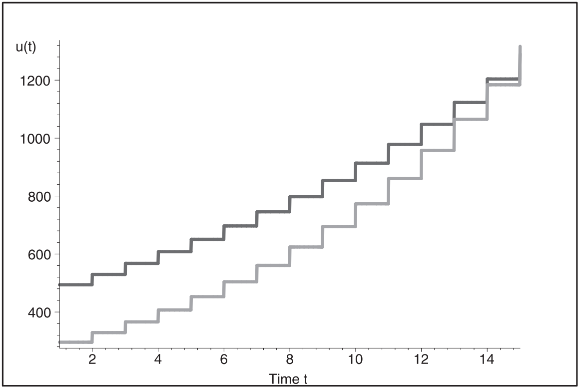

Fig. 2. Players' strategies: cooperation and Nash equilibrium

Modelling

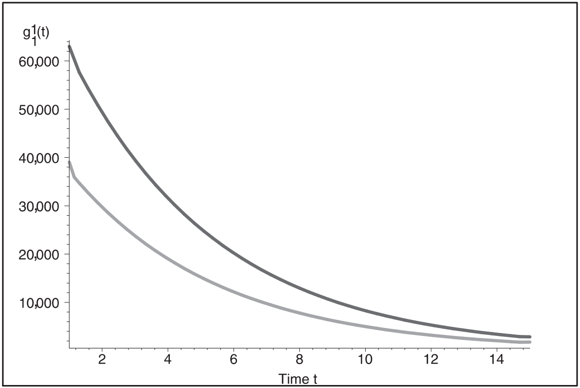

Fig. 3. Instantaneous payoffs: cooperation and Nash equilibrium

Modelling

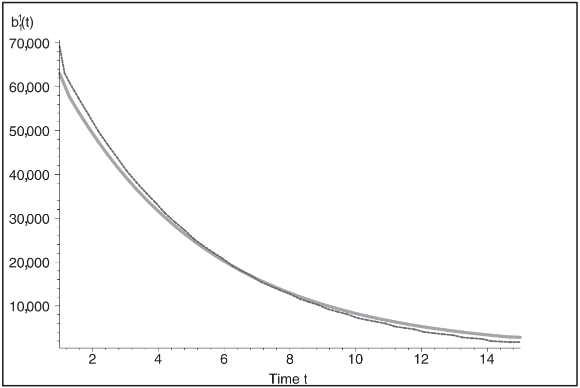

Fig. 4. Instantaneous payoffs and PDP

Conclusions

The dynamic, discrete-time game model where players exploit a common resource and have different criteria to optimize is investigated.

First, we construct the guaranteed payoffs and multicriteria Nash equilibrium.

Second, we present an approach to construct the cooperative equilibrium that guarantees the fulfillment of individual rationality conditions.

Third, we adopt dynamic stability concept for dynamic multicriteria games and construct payoff distribution procedure.

Fourth, we adopt conditions for rational behavior for dynamic multicriteria games.

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Thank you for the attention!